Investigating the Relationship Between Puzzles and Learning in Mathematics

Jess Martin
Mills College
jemartin@umbc.edu

Amy Hurst
UMBC
amyhurst@umbc.edu

ABSTRACT

There have been many ways that teachers have tried to make mathematics more interesting for their students. Innovations include interactive television shows, a hybrid of word and math puzzles, hands on activities, and expression of mathematics through movement. Before trying to introduce a new way to make mathematics interesting, we wanted to first determine if there is a relationship between a person's ability to solve puzzles and their mathematic skills. Puzzles were chosen whose solution's could be arrived at through common sense or trial and error as well as through problem solving techniques that become crucial in complex mathematics and proof writing. The puzzles we chose ranged from permutation puzzles to Tangrams. In addition to learning what techniques people use to solve problems, we also asked them to take an optional mathematics evaluation to get an idea of their mathematic skills.

Keywords: mathematics, puzzles, Tangrams, problem solving

1. INTRODUCTION

Mathematics can be a very intimidating topic for students who have a hard time learning the material. It can also be hard for students to understand why they need to learn abstract concepts if they can't relate it to a problem in the real world. Teachers are constantly trying to come up with ways to keep their students engaged, but what if there were a fun, weekly challenge that helped students develop math skills in their spare time as well? Inspired by math circles and Vi Hart videos, we wanted to develop mathematical puzzle challenges that can be done in any home, but the first step was to determine what the relationship between puzzles and mathematical skills actually is and how we can use it to help teach.

Using puzzles and games to teach any subject can be a risky decision. Often times, educational games don't feel like games at all because the focus becomes more on the educational content than on game design. It is important to take a similar approach to designing educational games as is used in designing games for entertainment. Educational games should present the player with a set of goals, provide feedback, and should teach the player skills that are necessary to move forward [5]. In addition to learning game design basics, we also had to think about what makes something a puzzle. Scott Kim has defined a puzzle as something that is fun and has a right answer [4]. He goes on to say that there are several things that make a puzzle fun: they're novel, not too easy, not too hard, and they're tricky. These are all things we had to consider while choosing which puzzles to include.

Wanting to keep this project accessible for most homes in the future, we started with paper puzzles that anyone can make. With the rise of affordable 3D printers, like the MakerBot, we hope that people will be able to print their own puzzles each week, as well as trophies to mark their accomplishments. We also plan to develop a web community to allow students to compete and collaborate with each other.

2. RELATED WORK

Studies in related work focus on innovations to make mathematics more enjoyable for students. In each study, a new way to introduce mathematic concepts is developed and then tested on a group of students, often times in a classroom.

One of the earlier studies we found was a 1993 study [2] that uses an interactive television series, *The Adventures of Jasper*, to present students with real-life applications for mathematics. The show is presented in the classroom, and has occasional interruptions to allow the students to divide into groups to discuss solutions to the present problem. This approach was called "anchored instruction" because "a narrative, presented on videodisc, connects or anchors mathematical concepts to realistic, complex situations," thus tying the concepts to meaningful problems and outcomes. About a dozen episodes of The Jasper Series were produced to help introduce mathematical concepts as tools.

One study focused on body movements as a way to supplement mathematics curricula [6]. Rather than using activities for students to interact with, the students became an active part of the interaction. Students were given an opportunity to learn algebra and geometry concepts through experiments such as math angels (see Figure 1). The technology of computer vision was paired with a projector to allow students to investigate concepts like transformations of objects using images of their own bodies and movements. Through math propulsion, students were able to interact with mathematics in a fun, beautiful, and expressive way in contrast to the typical, rigid way of learning mathematics.

Figure 1. Superimposed stills enable students to explore geometric rotations through "math angels." Image from [6].

Another project [7] involved using a hybrid of word puzzles and mathematical puzzles to emphasize the idea that math is just another language to be learned. Their puzzles focused more on mathematical operations rather than simply using mathematical vocabulary words in a word puzzle. Clues to the hybrid crossword and jumble puzzles require certain math skills to solve; hexadecimal clues work well for these puzzles as some hexadecimal values are already represented with letters rather than numbers (see Figure 2). These puzzles were used as a part of the Virtual Engineering Sciences Learning Lab, an online virtual learning environment that uses games and activities to explain basic science and math concepts. VESLL can be found on the free 3D virtual world Second Life. The VESLL project is still under development, but an initial assessment workshop was held on the Loyola Marymount University campus in the summer of 2010. The overall responses to the activities were positive and participants found the puzzles to be fun and felt that they helped them improve their learning of the concepts.

Figure 2. Hexadecimal jumble puzzle. Image from [7].

While these projects were all very inspiring helped the brainstorming process tremendously, they mostly dealt with more basic mathematic concepts than we originally wanted to study. While looking through *Innovations in Teach Abstract Algebra*, we came across an article describing ways to teach group theory through permutation puzzles. This article included a number of permutation puzzles, one being Top Spin which ended up being one of the puzzles used in our study. Kiltinen describes how it can be useful to teach students just enough Abstract Algebra for them to be able to solve permutation puzzles. This approach provides a safe balance between teaching complex concepts and overwhelming students with too much information.

After reading through many papers detailing related work, we felt ready to come up with a solid group of puzzles for our own study. These projects seemed to support our hypothesis that puzzle activities truly are beneficial to mathematical education, but some required a lot of special equipment. Our project looked specifically at puzzles that were easy to reproduce using materials we already had, with the exception of Top Spin. While a laser cutter was used to cut most of our pieces, a Silhouette SD digital cutting tool is a more affordable and practical alternative for a household. The design choice to use puzzles that can be easily reproduced was essential to the idea that these puzzles be accessible to most households rather than being solely in-class activities for students.

3. PUZZLES

3.1 Tangram

A Tangram is an ancient Chinese puzzles consisting of seven geometric shapes: a square, two small triangles, one medium triangle, a rhombus, and two large triangles. The shapes of a Tangram set themselves adhere to strict rules; constraints are placed on both degrees of angles in the shapes as well as the area of the shapes. Angles must be 45˚, 90˚, and 135˚ and the area of each shape is directly related to the area of other shapes. For example, the area of the rhombus is equal to the area of the square as well as twice the area of a small triangle. There are thousands of designs that can be made with a single Tangram set. We chose designs that ranged from easy to solve to extremely difficult. We also chose to present the designs as partially solved and provide the participants with three different sets of pieces to choose from to finish the puzzle.

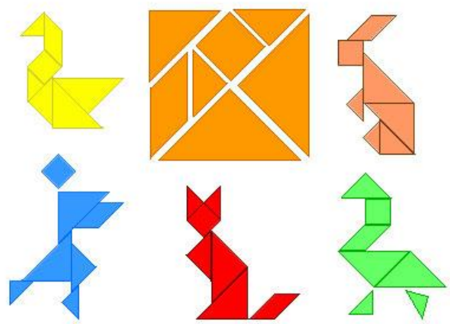


Figure 3. An example of a Tangram set and designs that can be made from it. Image from http://www.clccharter.org

We chose to use a variety of Tangram puzzles in our study to focus on geometry concepts including angles, congruency, similarity, and area. Transformations are also an essential concept used to solve a Tangram puzzle as often times the pieces need to be rotated to fit correctly into place. Designing the "wrong sets" meant looking carefully at the angles and total area needed to finish the puzzle and making sure that it could not be done with the pieces to be designed. It was our hopes that designing the wrong pieces in this way would lead the participants to eliminating them for the same reason.

3.2 Folding Puzzles

Two different folding puzzles were used as part of the study. The first puzzle was a 2-dimensional template of a 3-dimensional object. We used a pyramid template and asked participants to predict which edges would touch the edges of a marked face. After making a prediction, the participant could then fold the template into shape to see how correct they were. This also allowed them to understand where they went wrong if they incorrectly marked an edge or left one out. The second puzzle was a strip of paper marked like a gluing diagram of a topological surface. A gluing diagram is an assignment of a letter and an arrow to each edge of a polygon [1]. This allows us to represent surfaces using a simple diagram. For simplicity, we only used a diagram for a Möbius Strip and asked participants to predict what surface it represented. A Möbius Strip is a surface with just one side and can be thought of as a ring with a single half twist (see

Figure 4). For this task, if a participant wasn't familiar with Topology it would be hard to predict the surface so they were asked to describe it in any way they could.

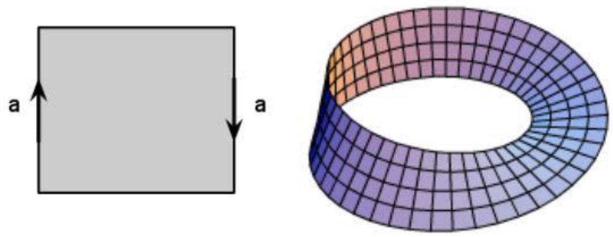


Figure 4. A gluing diagram of a Möbius Strip and the resulting surface. Image from [1].

This task challenged the participants to visualize a 3-dimensional object by simply observing its 2-dimensional counterpart. It also tested their knowledge of basic geometry concepts through counting edges. Each face of a pyramid will always have three edges of another face touching it when put together and this was something each participant had to consider, even if only briefly or intuitively. The Möbius Strip task tested the participant's knowledge of topological surfaces while forcing them to translate a 2-dimensional diagram into a surface they may or may not have known existed. If the participant correctly manipulated the diagram into a Möbius Strip, they were able to see for themselves that it is truly a one sided surface.

3.3 Tiling Puzzle

The tiling puzzle is an interesting example of an application of Mathematical Induction. For this task, an 8 x 8 checkerboard needs to be completely tiled so that only one square is uncovered. The participant is presented with the checkerboard that has one square covered with a 1 x 1 piece and the L shaped pieces they'll use to cover the rest of the board. Because there are many ways to cover the board this puzzle seems like an easy and straight forward task, but it can become very difficult in the end. The way you tile the pieces in the beginning determines whether or not you'll be able to fit all the pieces at the end.

By Mathematical Induction (MI), any checkerboard of dimension $2^n \times 2^n$, where *n* is a natural number (1, 2, 3,...), with one square removed can be tiled using L shaped pieces [8]. When using MI, you first show that the base case is true (n = 1), then by the inductive step, you assume that a $2^n \times 2^n$ board can be tiled to show that a $2^{n+1} \times 2^{n+1}$ board can be tiled. Once you've done this, you've actually proved that any board can be tiled because you've used an arbitrary *n* rather than a specific number. While this is the mathematical reasoning behind the problem, it doesn't provide a specific strategy that works. However, there is one important placement that will lead to a solution using the MI reasoning (see Figure 5).

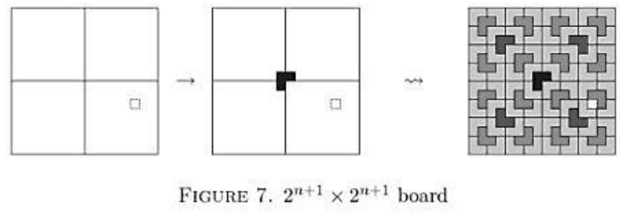


Figure 5. First cut a $2^{n+1} \times 2^{n+1}$ board into quarters (making four $2^n \times 2^n$ boards). Then, place an L shaped piece so that it covers the corner square of each quarter that isn't already missing a square. This results in four $2^n \times 2^n$ boards with one square removed each, which we know can be tiled by our inductive step. Image from [8].

3.4 Top Spin

The last puzzle that was used as part of this study was a permutation puzzle called Top Spin. This puzzle was the only puzzle that was purchased for the study. Because the puzzle has moving pieces, it was easiest to buy the puzzle and think of ways we could make our own similar puzzle in the future. Top Spin is an oval track with moving discs number 1 through 20 (see Figure 6). The object is to put the numbers in order by rotating the whole track left or right or by flipping four numbers at a time.

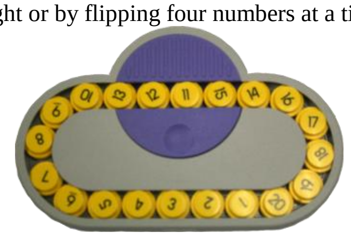


Figure 6. Top Spin 6 moves away from being solved. Rotating the whole track to the left or right one number is one move. Rotating the purple turn table once is also one

Permutation puzzles can be used to help teach group theory in Abstract Algebra. Specifically, these puzzles require relate to permutation groups in group theory and require basic concepts such as composition of permutations and inversion of permutations to solve. For the purposes of this study, we presented the participant with the puzzle just a few moves away from being solved. This setup narrowed down the number of possible permutations used to reach the correct answer. When you have a set of 6 elements, in this case moves, you can immediately find the number of possible permutations of that set by calculating $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. While this is the number of possible permutations for 6 elements, there will actually be less possibilities for solving the puzzle based on restrictions of moves. For example, if you had a set of 6 letters {a, b, c, d, e, f} one permutation could be switching the first and last letters: {f, b, c, d, e, a} but in Top Spin this isn't possible to do in just one move.

4. PILOT

Three participants took place in the pilot for our user study. The puzzle skills and mathematical skills of the three participants ranged from novice to advanced. Each participant was given one half solved Tangram puzzle, both folding puzzles, the tiling puzzle, and Top Spin to complete in a total of 30 minutes. Ten minutes were given for the Tangram, 10 minutes for the folding puzzles and tiling puzzle, and another 10 minutes for Top Spin. Upon completing the puzzles, or once the time was up, we then had an informal discussion with each participant about their math experiences that lasted about 10 minutes. Lastly, each participant was given an option to take a math evaluation. Only one participant in the pilot opted to skip the math evaluation.

4.1 Novice Participant

Our novice participant was able to correctly solve one of the puzzles, Top Spin, and opted out of the mathematic evaluation. This participant stated that she has always had a hard time with math and that puzzles actually make it more frustrating for her rather than making it more interesting. While none of the other participants were able to correctly solve the half solved Tangram, the novice participant was the only one to give up on it halfway through the allotted 10 minutes.

The novice participant's approach to each of the puzzles was to randomly guess and use trial and error. In the first folding task, she chose two arbitrary edges without trying to investigate the puzzle first. One of the guesses turned out to be a correct edge, but once she folded the pyramid into shape she took extra care to understand how the puzzle worked. It seemed like the first folding task influenced the participant's strategy in the second folding task. Rather than making the half twist to turn the strip of paper into a Möbius Strip, the participant excepted the final shape to be similar to a triangle and folded the paper in an accordion style to try to make the arrows meet.

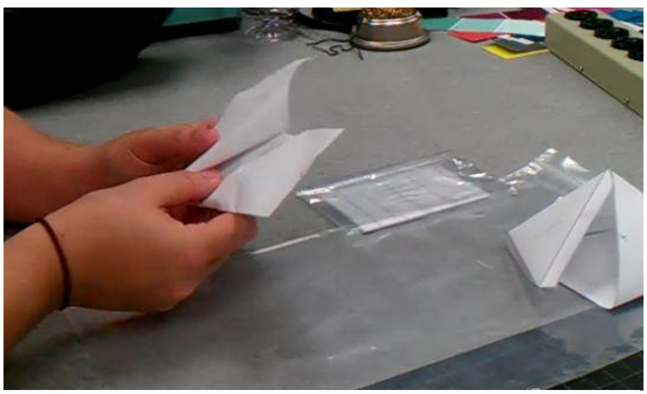


Figure 7. Novice participant working on the second folding puzzle.

In the tiling puzzle, each participant used a similar technique; at some point in the puzzle they each placed an L shaped piece around the 1 x 1 piece before proceeding. Like one of the advanced participants, this participant was not able to finish the tiling puzzle in the allotted five minutes. Both participants got to the very end and could not successfully put down their last two or three pieces.

4.2 Advanced Participants

We determined that two of our original participants were advanced based on both their puzzle performance and results on the math evaluation. Participant 1 was able to correctly solve four of the five puzzles and answered all of the questions on the math evaluation correctly with sufficient explanation. Participant 2 correctly solved two of the five puzzles and answered seven of the ten questions correctly on the math evaluation. One question on the second participant's math evaluation was left blank because he wasn't familiar with the word "derivative". Neither of the advanced participants were able to successfully solve the Tangram puzzle, but Participant 1 correctly solved the first folding puzzle while Participant 2 only chose two of the three correct edges. Participant 1 also finished the tiling puzzle but Participant 2 ran into problems at the end of the puzzle and was unable to complete it.

Both advanced participants correctly solved the second folding puzzle, but it was clear that Participant 1 had previous knowledge of Topological surfaces and Participant 2 did not. Immediately, Participant 1 knew how to fold the strip but he incorrectly called it a Torus (Topological name for the surface commonly referred to as a doughnut) before folding it into shape and correcting himself. While Participant 2 did not seem to be as familiar with Topology, he also knew exactly how to fold the shape but had a harder time describing it. Once the strip was folded into a Möbius Strip he had an easier time explaining it. He referred to it as being infinite and one sided. He also checked that he was correct in saying it was one sided by tracing the side of the surface with his finger.

Figure 8. Participant 2 tracing the Möbius Strip with his finger to make sure it's one sided.

Both our novice and advanced participants were able to solve Top Spin in a matter of seconds. Participant 2 and the novice participant did not talk through their thought process while solving Top Spin. Participant 1 took extra care to understand how to solve the puzzle before moving any of the discs. He explained that after not being able to solve the Tangram puzzle, he wanted to be sure he understood the problem before diving into it. His reasoning was that the first moves he made would impact the end result and he wanted to make sure the impact would be a positive one. This thought process was very similar to the problem solving technique of dividing a problem into multiple, easier to solve problems.

5. USER STUDY

Two things that really stood out from our pilot were how difficult it was for participants to solve the Tangram puzzle and how easy it was for them to solve Top Spin. This was actually the opposite of what we predicted, so after our first three participants we took time to reevaluate our puzzles.

Each of our three pilot participants acknowledged that if they had been able to place the Tangram pieces directly onto the design it would have been easier to solve. Our original setup forced the participant to mentally rescale their pieces, eliminate two sets of wrong pieces, and translate the correct pieces in such a way that they fit the design. We decided to take a scaffolding approach with our next participant and first provide Participant 4 with a Tangram design that was fully to scale (see Figure 9). She was able to solve the first Tangram quickly and moved on to the original Tangram puzzle from the pilot. Participant 4 was also able to solve the original Tangram in the allotted ten minutes. Her success led us to believe that providing participants with an easier puzzle first made the more advanced puzzle easier to solve.

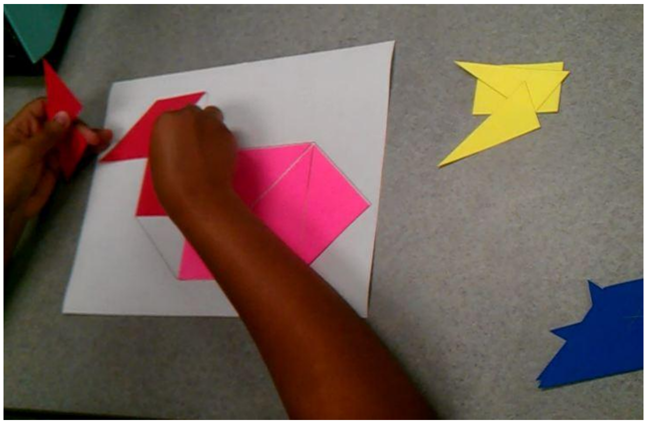


Figure 9. Participant 4 solving the fully scaled Tangram puzzle.

The second key change we made to the user study after the pilots was an adjustment to Top Spin. Originally, the puzzle was set up to be four moves away from being solved. Because some of the legal moves in Top Spin are rotating the whole track one space, this meant the participants in the study sometimes only had to think of two or three moves to solve it rather than four. Even though the puzzle was set up to be four moves away, a slight movement while learning how the puzzle works could easily reduce four moves to three. To remedy this, and to make sure the participants really had to perform all the moves expected, our setup for Participant 4 required six moves to solve. Participant 4 was able to correctly solve Top Spin in our study, but it took longer than any of the pilot participants.

A few pilot participants also suggested that Top Spin would have been more difficult if they weren't told how many moves they needed to perform. Some thought went into changing our procedure so the participants didn't know how many moves were needed but was decided against. In the future, we might require more moves to solve but there are no plans to present the puzzle without telling the participant how many moves are required to solve it.

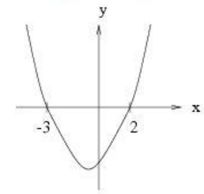
An interesting observation in Participant 4's session was during the second folding task. She approached the task in the same way as the novice participant. Participant 4 stated that she would be taking a Calculus course in the upcoming semester, so it is unlikely that she was previously familiar with Topological surfaces. These two outcomes led us to think about first introducing some Topological surfaces before presenting future participants with the gluing diagram. This way they will have some familiarity but will still have to visualize the gluing diagram in three dimensional space to determine the final surface.

6. MATH EVALUATION

Each participant was given the opportunity to complete an optional mathematical evaluation at the end of their session for an extra $5. Only our novice participant chose not to take the math evaluation. The evaluation was made of ten problems ranging from basic arithmetic to combinatorics and Calculus. Problems on the evaluation were multiple choice, word problems, and fill in the blank problems. We hoped to get a better understanding of each participants' mathematical skills through the evaluation in addition to what we were able to learn from our informal discussion. The results of our evaluation ranged from 100% to 50% correct.

If 5 is subtracted from 2.8 then the result is _______.

If f is a function whose graph is the parabola shown below, $f(x) < 0$ whenever
(a) $x > 0$
(b) $x > -3$
(c) $x < 2$
(d) $x < -3$ or $x > 2$
(e) $-3 < x < 2$



How many different ways can you rearrange the letters {a, b, c, d}?

Figure 10. Examples of questions from our mathematics evaluation

In the pilot sessions, Participant 1, the participant who correctly answered all of the questions on the math evaluation, also correctly solved more puzzles than the other pilot participants. Participant 4, the only participant not in the pilot, correctly answered half of the questions on the math evaluation and correctly solved four of the five puzzles, the same amount as Participant 1. It is interesting to note that while they were able to solve the same amount of puzzles, they were able to solve different puzzles. Participant 1 had no problem solving the second folding puzzle but Participant 4 did not end up with a Möbius Strip. Participant 4 was also the only participant to first have an easier Tangram to solve before the original bunny Tangram. She was able to solve both Tangram puzzles in the allotted ten minutes but Participant 1 was unable to solve the original puzzle.

Participant 2 also took part in the mathematical evaluation, answering 7 out of 10 questions correctly. His results on the math evaluation match his success with the puzzles in that he was able to solve 2 of the 5 puzzles, placing his performance at about average. Our novice participant, Participant 3, was the only one who opted out of the math evaluation. Because she's always had a hard time with math and was easily frustrated by the puzzles she decided not to take the evaluation because she felt it was likely she wouldn't know any of the answers.

7. FUTURE WORK

This project served as a great way to understand what puzzles work for our purposes as well as how we can intervene and provide hints when users get stuck. This work was very important but is just the beginning of our full project. We plan to continue working on this project as part of a Master's Thesis and ultimately hope to develop a weekly puzzle challenge. The next step of this process will be to fine-tune our compilation of puzzles and hints. We will do this by taking into account any suggestions made by the participants of our pilot and user study as well as their performance in the study. Similarly, we will also reevaluate our math evaluation to ensure that the questions asked are sufficient in determining a general idea of where participants stand in terms of their mathematical skills. Once we have made adjustments to our puzzles, hints, and math evaluation we will recruit more participants to continue with this study. At the minimum we would like to bring in 20 more participants to take part in this initial study before we move on to the next stage.

The second step will be to begin to gather more puzzles and activities that cover a wider range of mathematical concepts. As in this study, we plan to focus on puzzles and activities that can be easily reproduced in most homes. With a wider range of mathematical concepts we will be able to develop somewhat of a curriculum to be explored through our puzzles. During this stage we will also start working towards our ultimate goal by designing appropriate trophies that can be printed by users with a 3D printer. The last part of this stage will be to begin developing a shell for our eventual online community. Further in the future we would like to also develop a mobile app to accompany our puzzle challenges and online community.

8. ACKNOWLEDGEMENTS

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9. REFERENCES

[1] Averett, M. (2011). Topology. *Berkeley Math Circle Notes* (pp. 1-16).

[2] Barron, B., & Kantor, R. J. (1993). Tools to enhance math education: the Jasper series. *Communications of the ACM, 36*(5), 52-54. New York: ACM. DOI = http://doi.acm.org/10.1145/155049.155060.

[3] Kiltinen, J. O. Learning permutation group theory via puzzles. in Hibbard, A. C. and Maycock, E. J. *Innovations in Teaching Abstract Algebra,* Mathematical Association of America (Inc.), 2002, 131-136.

[4] Kim, S. What is a Puzzle?. in Fullerton, T. *Game Design Workshop: A Playcentric Approach to Creating Innovative Games,* Morgan Kaufman Publisher, Burlington, MA, 2008, 35-39.

[5] Linehan, C., Kirman, B., Lawson, S., & Chan, G. (2011). Practical, appropriate, empirically-validated guidelines for designing educational games. *Proceedings of the 2011 annual conference on Human factors in computing systems - CHI '11* (pp. 1979 - 1988). New York, New York, USA: ACM Press. DOI= http://doi.acm.org/10.1145/1978942.1979229.

[6] Mickelson, J., & Ju, W. (2011). Math propulsion. *Proceedings of the fifth international conference on Tangible, embedded, and embodied interaction - TEI '11* (p. 101). New York, New York, USA: ACM Press. DOI = http://doi.acm.org/10.1145/1935701.1935722.

[7] Neyer, A., August, S. E., & Hammers, M. (2011). Working together: words and math. *J. Comput. Small Coll., 26*(4), 197–203. USA: Consortium for Computing Sciences in Colleges. Retrieved from http://portal.acm.org/citation.cfm?id=1953573.1953606.

[8] Stankova, Z. and Rike, T. A decade of the Berkeley Math Circle: the American experience. American Mathematical Society, 2008.