Knowledge Transfer in Markov Decision Processes

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What is a Markov Decision Process (MDP)?

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- A probability function \( P : S \times A \times S \mapsto [0, 1] \) denoted by \( P_{ss'}^a \)
- A reward function \( R : S \times A \mapsto \mathbb{R} \) denoted by \( R_s^a \)
A Simple Example

The goal of an MDP is to maximize the reward over time.
Solving MDPs

- The optimal value function $V^*$ of an MDP:

$$V^*(s) = \max_{a \in A} (R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a V^*(s'))$$

where $\gamma \in (0, 1)$ is a discount factor.
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• Suppose we have already solved a similar problem, can we use that knowledge to help solve this one?
Motivation

How well will a policy learned on the first MDP work on the second one?:

![Diagram of two MDP environments with start and goal states]
Policy Transfer

\[ M_1 = (S_1, A, \{P_{ss'}^a\}, \{R_s^a\}) \]
\[ M_2 = (S_2, A, \{Q_{tt'}^a\}, \{R_t^a\}) \]

2 MDPs which share the same set of actions.
Policy Transfer

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\[ M_1 = (S_1, A, \{ P^{a}_{ss'} \}, \{ R^a_s \}) \]
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• Suppose there is a mapping \( \rho : S_1 \mapsto S_2 \) which maps each state in \( M_1 \) to a corresponding one in \( M_2 \).
Policy Transfer

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- Suppose there is a mapping $\rho : S_1 \mapsto S_2$ which maps each state in $M_1$ to a corresponding one in $M_2$.

- Then a policy $\pi_1$ on $M_1$ naturally induces a policy on $M_2$ as follows:
  \[
  \pi_1(s, a) = \pi_2(\rho(s), a)
  \]
Policy Transfer

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3. How do we choose \( M_1 \) and \( M_2 \)?

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M_1 = (S_1, A, \{P_{ss'}^a\}, \{R_s^a\})
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- How do we choose \( M_1 \) and \( M_2 \)?

- And once we do, how close to optimal will \( \pi_2 \) be?
Finding a Similar MDP

To measure distance between MDPs, look at reward and probability distribution in corresponding states.

Bisimulation Metric:

$$d(s, t) = \max_{a \in A}(|R^a_s - R^a_t| + \gamma T_K(d)(P^a_s, Q^a_t))$$

where $T_K(d)$ is a probability metric known as the Kantorovich.

Translation: How different are the rewards, and what are the chances of transitioning to “close” states from these ones?

Ferns, N. et al., Metrics For Finite Markov Decision Processes, 2004
The Kantorovich Metric

Calculating the Kantorovich distance amounts to solving a minimum cost flow (a linear program):

\[ d(s_i, t_j) : \text{Cost of shipping one unit from } s_i \text{ to } t_j \]
\[ f(s_i, t_j) : d(s_i, t_j) \times [\text{no. units shipped along } s_i t_j] \]

Goal: Find the minimum cost of shipping the contents of the Supply nodes to the Demand nodes, according to their needs.

Very expensive to solve!!
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- Given $P^a_s$, take $n$ random samples $(X_1, X_2, \ldots X_n)$ for state $s$ under action $a$.
- Similarly, for $Q^a_t$ take $n$ random samples $(Y_1, Y_2, \ldots Y_n)$ for state $t$ under action $a$.

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- Given \( P_s^a \), take \( n \) random samples \((X_1, X_2, ...X_n)\) for state \( s \) under action \( a \).
- Similarly, for \( Q_t^a \) take \( n \) random samples \((Y_1, Y_2, ...Y_n)\) for state \( t \) under action \( a \).
- The Kantorovich distance is then efficiently approximated by

\[
T_K(d)(P_s^a, Q_t^a) = \min_{\sigma} \frac{1}{n} \sum_{k=1}^{n} d(X_k, Y_{\sigma(k)})
\]

where \( \sigma \) is some permutation of the indices. (This is still a linear program, but it is an easier one!)

The Similarity Calculation

The goal of the metrics presented above is to approximate the difference in value functions when a policy $\pi_1$ is transferred from one MDP to another. This depends on two things:

1. How close to optimal was $\pi_1$ to begin with?
2. How similar are the MDPs?

Our bound:

$$\|V^{\pi_2} - V_2^*\| \leq \frac{2}{1 - \gamma} \max_{s \in S_1} d(s, \rho(s)) + \frac{1 + \gamma}{1 - \gamma} \|V^{\pi_1} - V_1^*\|$$
Experimental Results

Our similar MDPs:

Actions: move North, move South, move East, move West
Experimental Results

Metric distances between corresponding states in the two MDPs:

Distance between states increases as the goal approaches.
Experimental Results

Estimated distances compared to actual difference in value functions:

The metric overestimates the difference in value functions.
Why the Big Discrepancy?

- The metric is computed independent of the policy we are trying to transfer
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- The metric is computed independent of the policy we are trying to transfer
- It assumes the policy which maximizes the difference in value functions
- What if we incorporate the policy we are trying to transfer into the distance measure?
The Fixed Policy Distance Measure:

\[ d(s_k, t_j) = \sum_{a \in A} |\pi_1(s_k, a) \cdot (R_{s_k}^a - R_{t_j}^a)| + \gamma \sum_{a \in A} (\pi_1(s_k, a) \cdot T_k(d)(P_{s_k}^a, Q_{t_j}^a)) \]
Pros and Cons of Fixed Policy Metric

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  ◦ Under a reasonable policy, the estimated distance appears to be much closer to the actual difference in value functions
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  - What is a reasonable policy?
Pros and Cons of Fixed Policy Metric

• Pros:
  ◦ Under a reasonable policy, the estimated distance appears to be much closer to the actual difference in value functions

• Cons:
  ◦ What is a reasonable policy?
  ◦ The new measure is no longer a metric
Summary

**What We Did**

- Used Bisimulation to measure the distance between MDPs
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- Determined through experiments (many of which are not shown here) that the bisimulation method was too pessimistic for our purposes
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What Next?

- Use the original metric, but with larger time-steps
- Modify the Fixed-Policy measure so that it retains some of the properties of a metric