Question 2: The one about ceilings

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Show that $\ulcorner(\sqrt{3}+\sqrt{7})^{2000}\urcorner$is divisible by 2^{2000}. Note: $\ulcorner x \urcorner$is the smallest integer $\geq x$.

Proof, Method 1: Let $A = (\sqrt{3}+\sqrt{7})^{2000}$.

$$\begin{aligned}
(\sqrt{3}+\sqrt{7})^{2000} &= [(\sqrt{3}+\sqrt{7})^2]^{1000} \\
&= (7+3+2\sqrt{21})^{1000} \\
&= 2^{1000}(5+\sqrt{21})^{1000} \\
(5+\sqrt{21})^{1000} &= [(5+\sqrt{21})^2]^{500} \\
&= (25+2\cdot 5\sqrt{21}+21)^{500} \\
&= (46+2\cdot 5\sqrt{21})^{500} \\
&= 2^{500}(23+5\sqrt{21})^{500} \\
\Rightarrow A &= 2^{1500}(23+5\sqrt{21})^{500}
\end{aligned}$$

$$\begin{aligned}
(23+5\sqrt{21})^{500} &= [(23+5\sqrt{21})^2]^{250} \\
&= (23\cdot 23+5\cdot 5\cdot 21+2\cdot 5\cdot 23\sqrt{21})^{250}
\end{aligned}$$

Notation blurb: by ...2, I mean an integer ending in 2, such as 952.

Note that $23\cdot 23 = ...9 = 1 mod 4$. Similarly, $5\cdot 5\cdot 21 = ...5 = 1 mod 4$ $\Rightarrow 23\cdot 23 + 5\cdot 5\cdot 21 = ...14 = 2 mod 4$. Thus, at most one factor of 2 can be factored out any such term

Therefore, $(23+5\sqrt{21})^{500}$ can be written as:

$$\begin{aligned}
(23+5\sqrt{21})^{500} &= (2m+2n\sqrt{21})^{250} \\
&= 2^{250}(m+n\sqrt{21})^{250}
\end{aligned}$$

where m,n are integers, such that $m = ...7$ and $n = ...5$.

Let $M = m+n\sqrt{21}$, henceforth referred to as an *M-term*.

$$
\begin{aligned}
M^2 &= (m+n\sqrt{21})^2 \\
&= m^2 + 21\cdot n^2 + 2\cdot m\cdot n\cdot\sqrt{21} \\
&= (...7)^2 + 21(...5)^2 + 2\cdot(...7)(...5)\sqrt{21} \\
&= ...9 + ...5 + 2(...5)\sqrt{21} \\
&= 2(...7) + 2(...5)\sqrt{21}
\end{aligned}
$$

Note that this is exactly of the form $m + n\sqrt{21}$. Thus, the square of an *M-term* is also an *M-term*.

From the last calculation

$$
\begin{aligned}
A &= 2^{1500}(23+5\sqrt{21})^{500} \\
&= 2^{1750}(m+n\sqrt{21})^{250} \\
&= 2^{1750}M_1^{250} \\
&= 2^{1750}\cdot 2^{125}\cdot M_2^{125} \\
&= 2^{1875}\cdot M_2^{124}\cdot M_2 \\
&= 2^{1875}\cdot 2^{62}\cdot M_3^{62}\cdot M_2 \\
&= 2^{1937}\cdot 2^{31}\cdot M_4^{31}\cdot M_2 \\
&= 2^{1968}\cdot M_4^{30}\cdot M_4\cdot M_2 \\
&= 2^{1968}\cdot 2^{15}\cdot M_5^{15}\cdot M_4\cdot M_2 \\
&= 2^{1983}\cdot M_5^{14}\cdot M_5\cdot M_4\cdot M_2 \\
&= 2^{1990}\cdot M_6^{6}\cdot M_6\cdot M_5\cdot M_4\cdot M_2 \\
&= 2^{1993}\cdot M_7^{2}\cdot M_7\cdot M_6\cdot M_5\cdot M_4\cdot M_2 \\
&= 2^{1994}\cdot (M_8\cdot M_7)\cdot (M\cdot M_5)\cdot (M_4\cdot M_2) \\
&= 2^{1994}\cdot 2^3\cdot M_9\cdot M_{10}\cdot M_{11} \\
&= 2^{1997}\cdot 2\cdot M_{12}\cdot M_{11} \\
&= 2^{1999}\cdot M_{13} \\
\ulcorner A\urcorner &= \ulcorner 2^{1999}\cdot M_{13}\urcorner = 2^{1999}\ulcorner M\urcorner \\
&= 2^{1999}\ulcorner(...7)+(...5)\sqrt{21}\urcorner \\
&= 2^{1999}[(...7)+(...5)\ulcorner\sqrt{21}\urcorner] \\
&= 2^{1999}[(...7)+(...5)(5)] \\
&= 2^{1999}(...2) \\
\ulcorner A\urcorner &= 2^{2000}(x) \qquad ,\ x\geq 1
\end{aligned}
$$

Regardless of what the value of x is, $\ulcorner A\urcorner = 2^{2000}(x)$ is divisible by 2^{2000}.

Method 2 (originally proved by Pralash Panangaden): Claim: $\ulcorner(\sqrt{3}+\sqrt{7})^{2n}\urcorner = (\sqrt{7}+\sqrt{3})^{2n} + (\sqrt{7}-\sqrt{3})^{2n}$.

Proof: Let $K(n) = (\sqrt{7}+\sqrt{3})^{2n}+ (\sqrt{7}-\sqrt{3})$. One must show that (1) $K(n)$ is an integer and (2) $K(n)$ is the correct ceiling.

(1) Applying the binomial theorem,

$$\begin{aligned} K(n) &= (\sqrt{7}+\sqrt{3})^{2n} + (\sqrt{7}-\sqrt{3})^{2n} \\ &= \sum_{k=0}^{\infty} \left(\begin{array}{c} 2n \\ k \end{array} \right) (\sqrt{7})^k (\sqrt{3})^{2n-k} + \left(\begin{array}{c} 2n \\ k \end{array} \right) (\sqrt{7})^k (-\sqrt{3})^{2n-k} \end{aligned}$$

If k is even, then $2n-k$ is even, and thus as$K(n) = \sum_{k=0}^{\infty} \left(\begin{array}{c} 2n \\ k \end{array} \right) (7)^{k/2} (3)^{n-k/2} + \left(\begin{array}{c} 2n \\ k \end{array} \right) (7)^{k/2} (3)^{n-k/2}$, which is obviously an integer. Similarly, if k is odd, $K(n) = \sum_{k=0}^{\infty} \left(\begin{array}{c} 2n \\ k \end{array} \right) (\sqrt{7})^k (\sqrt{3})^{2n-k} - \left(\begin{array}{c} 2n \\ k \end{array} \right) (\sqrt{7})^k (\sqrt{3})^{2n-k} = 0$. Thus, $K(n)$is an integer.

(2) We want to prove that $\ulcorner(\sqrt{7}+\sqrt{3})^{2n}\urcorner = K(n)$. Let $\alpha = (\sqrt{7}+\sqrt{3})^{2n}$ and $\beta = (\sqrt{7}-\sqrt{3})^{2n}$. Then, we must prove that $\ulcorner\alpha\urcorner = \alpha+\beta$.

Note that $(\sqrt{7}-\sqrt{3}) < 1$, and thus $\beta = (\sqrt{7}-\sqrt{3})^{2n} < 1$

$$\begin{array}{ccc} 0 < & \beta & < 1 \\ \alpha < & \alpha+\beta & < \alpha+1 \end{array}$$

From (1), $\alpha+\beta$ is an integer. From the definition of $\ulcorner\alpha\urcorner$(the smallest integer $\geq x$), and since $\alpha \in \mathbb{R} \Rightarrow \alpha+1 \in \mathbb{R}$, there is exactly one integer that satisfies $\alpha < x < \alpha+1$, namely $\ulcorner\alpha\urcorner$. Thus, $\ulcorner\alpha\urcorner = \alpha+\beta$.

Having shown that $K(n) = (\sqrt{7}+\sqrt{3})^{2n}+ (\sqrt{7}-\sqrt{3})^{2n} = \ulcorner(\sqrt{7}+\sqrt{3})^{2n}\urcorner$, we must show that $2^{2n}|\ K(n)$.

$$\begin{aligned} (\sqrt{7}+\sqrt{3})^2 \cdot K(n) &= (\sqrt{7}+\sqrt{3})^2 (\sqrt{7}+\sqrt{3})^{2n} + (\sqrt{7}+\sqrt{3})^2 \frac{(\sqrt{7}-\sqrt{3})^2}{(\sqrt{7}-\sqrt{3})^2} (\sqrt{7}-\sqrt{3})^{2n} \\ (\sqrt{7}+\sqrt{3})^2 \cdot K(n) &= (\sqrt{7}+\sqrt{3})^{2n+2} + 16 \cdot (\sqrt{7}-\sqrt{3})^{2n-2} \end{aligned}$$

Similarly,

$$(\sqrt{7}-\sqrt{3})^2 \cdot K(n) = 16 \cdot (\sqrt{7}+\sqrt{3})^{2n-2} + (\sqrt{7}-\sqrt{3})^{2n+2}$$

Adding the two relations, we obtain,

$$\begin{aligned} 20 \cdot K(n) &= 16 \cdot [(\sqrt{7}+\sqrt{3})^{2n-2} + (\sqrt{7}+\sqrt{3})^{2n-2}] + [(\sqrt{7}+\sqrt{3})^{2n+2} + (\sqrt{7}-\sqrt{3})^{2n+2}] \\ 20 \cdot K(n) &= 16 \cdot K(n-1) + K(n+1) \qquad (*) \end{aligned}$$

By induction, show that $2^{2n}|\ K(n)$:

1. Base Case: $n = 1$:
   $K(n+1) = (\sqrt{7}+\sqrt{3})^{2n+2} + (\sqrt{7}-\sqrt{3})^{2n+2} =$, and $2^2 = 4|\ 20 \Rightarrow$base case holds. Also note that $K(0) = 2$, and that $2^0 = 1|\ 2$.

2. Induction Step: Assume $2^{2n}|\ K(n)$.
   By (*), $K(n+1) = 20 \cdot K(n) - 16 \cdot K(n-1) = 4[5 \cdot K(n) - 4 \cdot K(n-1)]$.
   By the induction hypothesis, $2^{2n}|\ K(n)$ and $2^{2n-2}|\ K(n-1)$.Thus,

$$\begin{aligned} K(n+1) &= 2^2[5 \cdot k_1 \cdot 2^{2n} + 2^2 \cdot k_2 \cdot 2^{2n-2}] \\ K(n+1) &= 5 \cdot k_1 \cdot 2^{2n+2} + k_2 \cdot 2^{2n+2} \end{aligned}$$

$\Rightarrow 2^{2n+2}|\ K(n+1)$.Thus, by induction, $\forall n, 2^{2n}|K(n)$.