Question 3: The one about e

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$\forall d \in \mathbb{R}$, define the series: $a_m(0) = \frac{d}{2^m}$, $a_m(j+1) = a_m(j)^2 + 2a_m(j)$. Show that $lim_{n\to\infty} a_n(n) = e^d - 1$.

Proof:

$$
\begin{aligned}
a_m(1) &= a_m(0)^2 + 2a_m(0) \\
&= a_m(0)^2 + 2a_m(0) + 1 - 1 \\
&= [a(0)+1]^2 - 1 \\
a_m(2) &= a_m(1)^2 + 2a_m(1) \\
&= [(a_m(0)+1)^2 - 1]^2 + 2[a_m(0)+1)^2 - 1] \\
&= [a_m(0)+1]^4 - 2[a_m(0)+1]^2 + 1 + 2[a_m(0)+1]^2 - 2 \\
&= [a_m(0)+1]^4 - 1
\end{aligned}
$$

Let $p = a_m(0) + 1$. Then,

$$
\begin{aligned}
a_m(3) &= a_m(2)^2 + 2a_m(2) \\
&= (p^4 - 1)^2 + 2(p^4 - 1) \\
&= p^8 - 2p^4 + 1 + 2p^4 - 2 \\
&= p^8 - 1
\end{aligned}
$$

Claim: $a_m(n) = p^{2^n} - 1$.
Proof by induction:

1. Base case: k = 1. As shown above,
   $a_m(1) = [a(0)+1]^2 - 1 = p^2 - 1$. Base case holds.

2. Induction Hypothesis: Assume $a_m(k) = p^{2^k} - 1$.

$$
\begin{aligned}
a_m(k+1) &= a_m(k)^2 + 2a_m(k) \\
&= [p^{2^k} - 1]^2 + 2[p^{2^k} - 1] \\
&= p^{2^{k+1}} - 2p^{2^k} + 1 + 2p^{2^k} - 2 \\
a_m(k+1) &= p^{2^{k+1}} - 1
\end{aligned}
$$

$\Rightarrow$By induction, $\forall n$, $a_m(n) = p^{2^n} - 1$.

Thus,

$$\begin{aligned}
a_m(n) &= p^{2^n} - 1 \\
&= [a(0) + 1]^{2^n} - 1 \\
&= [\frac{d}{2^m} + 1]^{2^n} - 1 \\
lim_{n\to\infty} a_n(n) &= lim_{n\to\infty}[\frac{d}{2^m} + 1]^{2^n} - 1 \\
&= lim_{n\to\infty}[\frac{d}{t} + 1]^{t} - 1 \\
&= e^d - 1
\end{aligned}$$

Lemma: $lim_{x\to\infty}[\frac{a}{x} + 1]^x = e^a$

Proof:

Let $f(x) = ln(1 + ax)$.

It follows that

$$\begin{aligned}
f'(x) &= \frac{a}{1 + ax}, \\
f'(0) &= a
\end{aligned}$$

.

Since $\frac{ln(1+ax)}{x}$is continuous, then

$$lim_{x\to 0}\frac{ln(1 + ax)}{x} = f'(0) = a$$

But

$$\frac{ln(1 + ax)}{x} = \frac{1}{x}ln(1 + ax) = ln(1 + ax)^{1/x}$$

$$\begin{aligned}
lim_{x\to 0} ln(1 + ax)^{1/x} &= a \\
lim_{x\to 0} e^{ln(1+ax)^{1/x}} &= e^a \\
lim_{x\to 0}(1 + ax)^{1/x} &= e^a
\end{aligned}$$

Let $m = \frac{1}{x}$, then $lim_{m\to 0}(1 + \frac{a}{m})^m = e^a$.