Question 1: The one about term decomposition

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Show that any integer n $\geq$1 can be written as the sum of terms of the form $2^m 3^n$, such that no term divides any other term.

For example, $17 = 9 + 8 = 2^0 3^2 + 2^3 3^0$ is a valid decomposition, but $10 = 9 + 1 = 2^0 3^2 + 2^0 3^0$ is not (because $1 \mid 9$).

Proof: Any number n is either even or odd, i.e. can be written as $n = 2k$ or $n = 2k + 1$, for a $k \in \mathbb{N}$. We shall treat these 2 cases independently:

1. Let $n = 2k$. This case is trivial, as the representation of n will be of the form $n = 2 \cdot [representation\, of\; k]$. Since the 2 is factored out, it will be common to all the terms in the representation of k, and thus will not affect the divisibility of terms. Thus, we are only concerned in the representation of k, which can again be found by considering the 2 cases: k odd and k even. Recurse away!

2. Let $n = 2k+1$. We shall find the integer containing the greatest power of 3, i.e. $p = 3^m$. Thus,

$$n = 3^m + q$$

   It should be noted that if q is a power of 3, i.e. $q = 3^r$, it must be that $r < m$ (otherwise, $q$would have been considered the greatest). In addition, note that q is not a simple power of 3, but rather a product of a power of 2 and 3 (i.e. $q = 2^s \cdot 3^r$). Since $p = 3^m$is odd, and n is odd, it follows that q must be even, and thus must contain an even coefficient, i.e of the form 2^s. Therefore, since $q = 2^s \cdot 3^r$and $p = 3^m$, it follows that p is not divisible by q, and thus the initial condition is still satisfied. Continue by finding the representation of q, and using it in the representation of n, where the arguments above hold.