Question 4: The one about rings

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Let R be a ring, such that $\forall x. x^3 = x$. Prove that R is commutative.

Proof: A ring R is commutative if $\forall x, y. xy = yx$. For the sake of consistency, define multiplication to be$x \cdot y = xy$ rather than $x \cdot y = yx$. If needed, the distinction will be made by explicitly using the terms left (the former) or right (the latter) multiplication.

Then,

$$\begin{aligned}
(x+y)^3 &= (x+y)(x+y)^2 \\
&= (x+y)(x^2 + xy + yx + y^2) \\
&= x^3 + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y^3 \\
&= x + y + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x
\end{aligned}$$

But $(x+y)^3 = x + y \Rightarrow$

$$x^2y + xyx + xy^2 + yx^2 + yxy + y^2x = 0 \tag{1}$$

Similarly,

$$\begin{aligned}
(x-y)^3 &= (x-y)(x-y)^2 \\
&= x - y - x^2y - xyx + xy^2 - yx^2 + yxy + y^2x
\end{aligned}$$

Since $(x-y)^3 = x - y \Rightarrow$

$$-x^2y - xyx + xy^2 - yx^2 + yxy + y^2x = 0 \tag{2}$$

Subtracting (2) from (1),

$$2xyx + 2yx^2 + 2x^2y = 0 \tag{3}$$

Substituting $y = x$ in (3),

$$\begin{aligned}
6x^3 &= 0 \\
6x &= 0
\end{aligned} \tag{4}$$

To simplify further calculations, it should be noted that $x^4 = x \cdot x^3 = x^2$.
On the other hand,

$$\begin{aligned}
(x+x^2)^3 &= (x+x^2)(x+x^2)^2 \\
&= (x+x^2)(x^2+2x^3+x^4) \\
&= (x+x^2)(2x^2+2x) \\
&= 2x^3+2x^2+2x^4+2x^3 \\
x+x^2 &= 4x+4x^2 \\
3x+3x^2 &= 0 \\
3x^2 &= -3x
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
(x-x^2)^3 &= (x-x^2)(x-x^2)^2 \\
&= (x-x^2)(2x^2-2x) \\
x-x^2 &= 4x-4x^2 \\
3x^2 &= 3x
\end{aligned} \tag{6}$$

From (5) and (6),

$$3x = -3x = 3x^2 \tag{7}$$

Using (7),

$$\begin{aligned}
3(x+y)^3 &= 3(x+y) \\
3(x+y)^2(x+y) &= 3x+3y \\
3(x+y)(x+y) &= 3x^2+3y^2 \\
3(x^2+xy+yx+y^2) &= 3x^2+3y^2 \\
3xy+3yx &= 0 \\
3xy &= -3yx
\end{aligned} \tag{8}$$

Since $3x = -3x$, it follows that $3xy = -3xy$. From (8), $3xy = -3yx$, and thus

$$\begin{aligned}
3xy &= -1(-3yx) \\
3xy &= 3yx
\end{aligned} \tag{9}$$

In order to show that R is commutative, one must show that $xy = yx$. This would follow from (9)if one could show that $2xy = 2yx$.

By multiplying (3) twice, i.e. left and right multiplication, one obtains:

$$
\begin{aligned}
2 \cdot x \cdot (xyx + yx^2 + x^2y) &= x \cdot 0 \\
2(x^2yx + xyx^2 + x^3y) &= 0 \\
2(x^2yx + xyx^2 + xy) &= 0 \qquad (10)
\end{aligned}
$$

As well as

$$
\begin{aligned}
2(xyx + yx^2 + x^2y) \cdot x &= 0 \cdot x \\
2(xyx^2 + yx^3 + x^2yx) &= 0 \\
2(xyx^2 + yx + x^2yx) &= 0 \qquad (11)
\end{aligned}
$$

Subtracting (11) from (10),

$$
\begin{aligned}
2(x^2yx + xyx^2 + xy) - 2(xyx^2 + yx + x^2yx) &= 0 \\
2xy - 2yx &= 0 \\
2xy &= 2yx \qquad (12)
\end{aligned}
$$

Thus, subtracting (12) from (9) it follows that:

$$
\begin{aligned}
3xy - 2xy &= 3yx - 2yx \\
xy &= yx
\end{aligned}
$$

Since $\forall x, y. xy = yx$, the ring R is commutative.